1 Abstract

In this paper we propose an algorithm for the optimal bit allocation among dependent quantizers for the minimum maximum (MINMAX) distortion criterion. We compare this algorithm to the well known Lagrange multiplier method for the minimum average (MINAVE) distortion criterion. We point out the differences between these two distortion criteria, and their implications for coding applications. We argue that even though the MINAVE criterion is more popular, in many cases, the MINMAX criterion is more appropriate. We introduce the algorithms for solving the optimal bit allocation problem among dependent quantizers for both criteria and highlight the similarities and differences. We present the two algorithms in the same framework, which sheds new light on the known algorithm for the MINAVE criterion and connects it with the proposed algorithm for the MINMAX criterion. Furthermore, we point out that any problem which can be solved for the MINAVE criterion, can also be solved for the MINMAX criterion, since both approaches are based on the same assumptions. Finally we present results of a still frame compression scheme, which is based on the Intra frame encoding of H.263.

2 Introduction

There exists an inherent tradeoff between the rate and the distortion of a lossy compression scheme. One common approach to mathematically formulate this relationship is to minimize the average (total) distortion for a given bit rate, or vice versa, to minimize the bit rate for a given average distortion. In other words, the MINAVE criterion is employed. The philosophy behind this approach is that if the average (total) distortion is minimized then, in the long run, the best quality is obtained.

It is well known [1, 2, 3, 4] that the Lagrangian multiplier method is well suited for these kind of constrained optimization problems. It converts the "hard" constrained problem into a set of "easy" unconstrained problems, parameterized by the Lagrange multiplier λ. In the case of dependent quantizers, these unconstrained problems can be solved efficiently by dynamic programming (DP). The optimal solution is found by using a search, for example the bisection method, to find the optimal λ^* which in turn results in an unconstrained problem using exactly the number of bits available.

It is interesting to notice that with this popular approach, a large variability among the different source distortions is possible. When the sources are consecutive in time or space, such as different frames of a sequence or different blocks in a frame, this variability in quality can be very disturbing and the perceived quality is low even though the average distortion is minimized.

A different approach to formalize the relationship between the rate and the distortion is the minimum maximum distortion approach, where the goal is to minimize the maximum source distortion for a given bit rate, or vice versa, to minimize the bit rate for a given maximum source distortion. In other words, the MINMAX criterion is employed. The philosophy behind this approach is that by minimizing the maximum source distortion, no single source distortion will be extremely bad and hence the overall quality will be quite constant. In fact, the MINMAX criterion is an ideal choice, when the goal is to achieve an almost constant distortion which is as small as possible for the available bit rate.

In the literature, the MINAVE criterion is much more common than the MINMAX. This is mostly due to the fact that efficient algorithms have been available for the MINAVE criterion, while such algorithms have been lacking for the MINMAX criterion.

The MINMAX problem for independent quantizers has been studied in [5] and a simple algorithm has been proposed. In this paper we study a scheme which finds the optimal solution for the MINMAX criterion for depended quantizers, where the dependency can be arbitrarily large.

First we show that quantizer selection for the minimum bit rate for a given maximum source distortion can be found in a dependent coding framework using DP. Then we employ the bisection algorithm to minimize the maximum source distortion for a given bit rate. This iterative algorithm invokes the above mentioned DP scheme and we prove that it converges to the optimal solution.

In section 3 we introduce the notation and assumptions and formulate the problem mathematically. In section 4 we introduce a toy example which we will use to demonstrate the two different algorithms. In section 5 we review the Lagrangian multiplier method for dependent quantizers. In section 6 we introduce an efficient algorithm for the optimal bit allocation among dependent quantizers for the MINMAX criterion. In section 7 we apply both algorithms to the Intra frame encoding scheme used in H.263 and compare the different results with respect to the mean and variance of the resulting distortion. Finally in section 8 we summarize the paper and draw our conclusions.

3 Notation, Assumptions and Problem Formulation

In this section we introduce the necessary notation, the underlying assumptions and the mathematical formulation of the optimal bit allocation problem. In this paper, we are concerned about the optimal bit allocation among sources in a dependent coding framework. As shown in Fig. (1) a dependent coding framework implies that the source rate $r_i(x_{i-a}, \ldots, x_{x+b})$ and/or source distortion $d_i(x_{i-a}, \ldots, x_{x+b})$ for a given source S_i depends not only on the quantizer x_i applied to that source, but also on neighboring quantizers $x_{i-a}, \ldots, x_{x+b}$ in a neighborhood defined by two non-negative integers a and b. Example for such dependent coding frameworks are all predictive coding schemes, such as motion compensated video compression, segmentation encoding, image coding, etc. For example, in motion compensated video compression, the quantizer selected for the previous frame has a direct influence on the rate-distortion characteristic of the current frame, since the reconstructed previous frame and the motion information is used to predict the current frame.

The total rate for encoding all sources $R(x_0, \ldots, x_{N-1})$ is the sum of the source rates, and defined as follows,

$$R(x_0, \ldots, x_{N-1}) = \sum_{i=0}^{N-1} r_i(x_{i-a}, \ldots, x_{i+b}). \tag{1}$$

Depending on the employed distortion criterion, the distortion for encoding all sources $D(x_0, \ldots, x_{N-1})$ is the sum (or average) of the source distortions (MINAVE),

$$D(x_0, \ldots, x_{N-1}) = \sum_{i=0}^{N-1} d_i(x_{i-a}, \ldots, x_{i+b}), \tag{2}$$

or the maximum of the source distortions (MINMAX),

$$D(x_0, \ldots, x_{N-1}) = \max_{i \in [0, \ldots, N-1]} \{d_i(x_{i-a}, \ldots, x_{i+b})\}. \tag{3}$$

In either case, two optimal bit allocation problems can be formulated, the minimum rate problem and the minimum distortion problem (which is also called the rate constraint problem). In the minimum rate problem we are looking for the quantizer sequence which results in the smallest bit rate for a given maximum distortion. This can be formulated as follows,

$$\min_{x_0, \ldots, x_{N-1}} R(x_0, \ldots, x_{N-1}) \text{ s.t.: } D(x_0, \ldots, x_{N-1}) \leq D_{max}, \tag{4}$$

where D_{max} is the largest permissible distortion.

In the minimum distortion problem we are looking for the quantizer sequence which results in the smallest distortion for a given maximum bit rate. This can be formulated as follows,

$$\min_{x_0, \ldots, x_{N-1}} D(x_0, \ldots, x_{N-1}) \text{ s.t.: } R(x_0, \ldots, x_{N-1}) \leq R_{max}, \tag{5}$$

where R_{max} is the maximum number of bits available. Note that the above formulations hold for either of the two distortion criterions.

4 Toy Example

In this section we introduce a toy example which we will use to illustrate the concepts discussed in the remainder of the paper. This example represents the simplest case of a dependent coding framework, but it contains all the elements of a more complex (and realistic) scheme.

We consider the problem of coding two consecutive image blocks, which represent the sources S_0 and S_1. The set of admissible quantizers for block 0 is $X_0 = \{1, 2, 3\}$, and the set of admissible quantizers for block 1, X_1 is the same as X_0. We use two bits to encode x_0. If x_1 equals x_0 then only one bit is sent, otherwise two bit are sent, indicating which of the other two admissible quantizers is used. To further simplify the example, we exclude quantizer 3 from the admissible quantizer sets, but maintain the above predictive quantizer encoding scheme. Note that the above predictive quantizer encoding scheme results in a dependent coding framework, with a neighborhood of one ($a = 1$, $b = 0$).

The graph on the left side in Fig. 2 represents the above example, where S indicates the start and T the termination of the graph. In the left graph, the numbers above the edges represent the number of bits used to encode the quantizers, which is two if the quantizer changes and one otherwise. The nodes, which are the intersection between the quantizers and the blocks, have a box associated. The rate for encoding the specific block using the specific quantizer is written in the box, above the node and the distortion which occurs when the specific block is encoded using the specific quantizer is written inside of the box, below the node. Note that we assume that quantizer two is coarser than quantizer one and hence the distortion for quantizer two is higher but its rate lower.

The graph on the right side in Fig. 2 is equivalent the graph on the left side, but it is in the notation we introduced in the preceding section. We simply merged the node rate and distortion into the transition rate and distortion. In other words, the number inside the box, above the edge, represents the rate $r_i(x_{i-1}, x_i)$ and the number inside the box, below the edge, represents the distortion $d_i(x_{i-1}, x_i)$.

The goal is now to find that quantizer sequence which is the optimal solution to the minimum rate or the minimum distortion problem for either of the distortion criterions. Clearly we want to find this sequence not using an exhaustive search, which in this case means, without testing all of the four possible quantizer sequences.

5 The MINAVE Criterion

In this section we show how the optimal bit allocation problem can be solved for the MINAVE criterion which is defined in Eq. (2). The solution is based on the Lagrange multiplier method and dynamic programming. Note that for the MINAVE criterion, the total rate and the total distortion are of exactly the same form and hence the solution approach for the minimum rate problem is equivalent to the solution approach for the minimum distortion problem. Therefore, in this section, we will concentrate on the minimum distortion problem, keeping in mind that by re-labeling the function names, i.e., $r(\cdot) \leftarrow d(\cdot)$, $d(\cdot) \leftarrow r(\cdot)$, $R(\cdot) \leftarrow D(\cdot)$, and, $D(\cdot) \leftarrow R(\cdot)$, the minimum rate problem can be solved using the same approach.

The basic idea behind the Lagrange multiplier method is to merge the rate and the distortion with a Lagrangian multiplier λ. This results in the Lagrangian cost function which is of the following form,

$$J_\lambda(x_0, \ldots, x_{N-1}) = D(x_0, \ldots, x_{N-1}) + \lambda \cdot R(x_0, \ldots, x_{N-1}). \tag{6}$$

The goal of the Lagrange multiplier method is to convert the "hard" constrained problem of Eq. (5) into a set of "easy" unconstrained problems parameterized by λ.

It has been shown in [1, 2] that if there is a λ^* such that,

$$[x_0^*, \ldots, x_{N-1}^*] = \arg \min_{x_0, \ldots, x_{N-1}} J_{\lambda^*}(x_0, \ldots, x_{N-1}), \tag{7}$$

leads to $R(x_0^*, \ldots, x_{N-1}^*) = R_{max}$, then $[x_0^*, \ldots, x_{N-1}^*]$ is also an optimal solution to the minimum distortion problem of Eq. (5) for the MINAVE criterion. It is well known that when λ sweeps from zero to infinity, the solution to problem (7) traces out the convex hull of the operational rate distortion curve, which is a non-increasing function. Hence bisection [6] can be used to find λ^*. The main problem with the Lagrange multiplier method is that only solutions which belong to the convex hull can be found. As we will see, using the toy example, there are also optimal solutions above the convex hull, which are inaccessible.

Clearly the efficiency of the Lagrange multiplier method depends on the assumption that the unconstrained problem of Eq. (7) can be solved efficiently. Based on the assumptions made in section 3, the source rates and distortions only depend on the quantizers selected in a neighborhood around the current source. Therefore the Lagrangian cost function can be expressed as follows,

$$J_\lambda(x_0, \ldots, x_{N-1}) = \sum_{i=0}^{N-1} (d_i(x_{i-a}, \ldots, x_{i+b}) + \lambda \cdot r_i(x_{i-a}, \ldots, x_{i+b})). \tag{8}$$

Hence it is always possible to solve the unconstrained problem efficiently using dynamic programming, where the efficiency of the solution directly depends on the size of the neighborhood $(a + b)$ [4].

5.1 Toy Example

Having described the general approach for the optimal bit allocation among dependent quantizers for the MINAVE criterion, we apply the algorithm to our toy example. First, we check the extreme cases for λ in Eq. (6). If we let λ converge to infinity, then the rate dominates the cost function and the quantizer sequence which results in the smallest rate will be selected. Since quantizer two is the coarsest quantizer available, the resulting quantizer sequence is $\{2, 2\}$. We call this sequence (A) and its rate and distortion are displayed in Table 1. If we let λ converge towards zero, then the distortion dominates the cost function and the quantizer sequence which results in the smallest distortion will be selected. Since quantizer one is the finest quantizer available the resulting quantizer sequence is $\{1, 1\}$. The rate and distortion for this sequence, which we call sequence (B), is displayed in Table 1.

After having established the extreme cases, we consider a more interesting case, where the rate and the distortion are balanced by setting λ equal to one. Now we cannot solve the unconstrained problem just by inspection as we did before, but we use dynamic programming to find the shortest path from S to T in Fig. 3. In this figure, we merged the rate and distortion from Fig. 2 using a λ of one. In the boxes we wrote down the optimal Lagrangian cost up to and including the node, and the dashed arrows indicate the backpointers used to remember the optimal path. As can be seen from this figure, the shortest path trough the graph for $\lambda = 1$, is $\{1, 2\}$, resulting in a Lagrangian cost of 21, a rate of 13 and a distortion of 8. We call this sequence (C) and its rate and distortion are displayed in Table 1.

Since there are two blocks and each block can be quantized using two different quantizers, there are four possible quantizer sequences. The only quantizer sequence we have not discussed yet is $\{2, 1\}$ which we call (D). Again, we recorded the rate and distortion of sequence (D) in Table 1. Having the rate and distortion of all possible quantizer sequences, we can draw the operational rate distortion curve, which is displayed in Fig. 5.

We go now through the exercise of finding the solution to the minimum distortion problem of Eq. (5), where the available bits R_{max} are set to 18. Since we use bisection to find the optimal λ^*, we need an initial bracket for λ, say $\lambda_l = 0.1$ and $\lambda_u = 10$. We then build the graph for λ_l and find the shortest path which results in the sequence (B) (this can be seen from Fig. 5). Then we repeat the same for λ_u. This results in sequence (A). Since the rate for (B) is higher than 18 and the rate for (A) is lower than 18, we know that the optimal λ^* has to be between λ_l and λ_u. We then pick a λ_m which lies between λ_l and λ_u. One can simple select λ_m to be the average of the bracketing λ's or one can use some additional knowledge to speed up the iteration, like the fact that the slope of the convex hull and λ are related as follows, slope = -1/λ [7, 8, 4]. We then build the graph for λ_m and find the shortest path which then results in a certain rate and distortion. If the rate is higher than 18 then $\lambda_l = \lambda_m$ otherwise $\lambda_u = \lambda_m$. Then a new λ_m is selected and the iteration is repeated until the optimal solution is found or the λ bracket is small enough or a certain maximum number of iterations have been carried out.

By inspecting Fig. 5 it is clear that the final solution is (C), since the true optimal solution (D) does not belong to the convex hull. This highlights a shortcoming of the Lagrange multiplier method, that if the operational rate distortion function is not convex, then not all optimal solutions can be found. These solutions are called inaccessible. Fig. 5 clearly shows that operational rate distortion functions are not necessarily convex.

6 The MINMAX Criterion

In this section we propose a general algorithm for the optimal bit allocation among dependent quantizers for the MINMAX criterion [9]. As in the previous section, we first introduce the general algorithm and then apply it to the toy example.

The basic idea behind the proposed algorithm is to solve the minimum rate problem optimally using dynamic programming. This is possible since the maximum distortion constraint D_{max} applies to *each* source and not to the sum of the source distortions. We then prove that the operational rate distortion function is non-increasing. Therefore we can solve the minimum distortion problem, which is a min max problem, using bisection, where in each bisection iteration the minimum rate problem is solved using a different D_{max}.

6.1 The Minimum Rate Problem

In this section, we solve the minimum rate problem which is described in Eq. (4) for the MINMAX criterion. The key observation for the derivation of the optimal solution is, that the maximum distortion D_{max} constraint applies to each source, and not, as in the case of the MINAVE criterion, to the sum of the source distortions. We can make use of this fact by redefining the source rates as follows,

$$r_i(x_{i-a}, \ldots, x_{i+b}) = \begin{cases} \infty & : \quad d_i(x_{i-a}, \ldots, x_{i+b}) > D_{max} \\ r_i(x_{i-a}, \ldots, x_{i+b}) & : \quad d_i(x_{i-a}, \ldots, x_{i+b}) \leq D_{max} \end{cases} . \tag{9}$$

In words, the rate for a source with a distortion which is larger then the maximum permissible distortion is set to infinity. This results in the fact that, given a feasible solution exists, the quantizer sequence which minimizes the total rate, as defined in Eq. (1), will not result in any source distortion greater than D_{max}. If no feasible solution exists, then the resulting minimum total rate is infinite, hence this situation can easily be detected and D_{max} can be increased. In other words, the minimum rate problem, which is a constrained optimization problem, can be transformed into an unconstrained optimization problem using the above re-definition of the source rates.

The structure of the total rate formula in Eq. (1) is equivalent to the structure of the Lagrangian cost function for the MINAVE case in Eq. (8). Hence the optimal solution to the unconstrained minimum rate problem can also be solved by dynamic programming. For an explicit algebraic derivation of this fact, please see [4, 9].

We can now calculate the operational rate distortion function $R^*(D_{max})$ as follows,

$$R^*(D_{max}) = \min_{x_0,\ldots,x_{N-1}} R(x_0,\ldots,x_{N-1}) \text{ s.t.: } D(x_0,\ldots,x_{N-1}) \leq D_{max}, \quad (10)$$

where we assume that D_{max} is a variable.

6.2 The Minimum Distortion Problem

The proposed optimal bit allocation algorithm for the minimum distortion problem is based on the fact that we can optimally solve the minimum rate problem. In other words, for every given D_{max} we can find the quantizer sequence which results in $R^*(D_{max})$, the minimum rate for encoding the combined sources, where each source distortion has to be below the maximum distortion D_{max} (see Eq. (10)). We use the following Theorem to formulate an iterative procedure to find the optimal solution for the minimum distortion problem.

Theorem 1 *$R^*(D_{max})$ is a non-increasing function of D_{max}.*

Proof: Let $D^2_{max} \geq D^1_{max}$, $[{}^1x^*_0,\ldots,{}^1x^*_{N-1}]$ be the optimal solution of Eq. (4) for $D_{max}{=}D^1_{max}$, and $[{}^2x^*_0,\ldots,{}^2x^*_{N-1}]$ the optimal solution of Eq. (4) for $D_{max}{=}D^2_{max}$. Since $D^1_{max} \leq D^2_{max}$, $[{}^1x^*_0,\ldots,{}^1x^*_{N-1}]$ is a possible solution of Eq. (4) for $D_{max}{=}D^2_{max}$, using $R^*(D^1_{max})$ bits. Since $[{}^2x^*_0,\ldots,{}^2x^*_{N-1}]$ is the optimal solution of Eq. (4) for $D_{max}{=}D^2_{max}$, it follows that $R^*(D^2_{max}) \leq R^*(D^1_{max})$. ■

The above Theorem is intuitively clear since it simply states that if a greater maximum distortion is permissible, then we should be able to encode the sources with a smaller number of bits. Note that even though this seems obvious, this only holds true because we can solve the minimum rate case optimally.

Having shown that $R^*(D_{max})$ is a non-increasing function, we can use bisection to find the optimal D^*_{max} such that $R^*(D^*_{max}) = R_{max}$, which solves the minimum distortion problem of Eq. (5) for the MINMAX criterion.

6.3 Toy Example

In this section we apply the above general theory for the optimal bit allocation among dependent quantizers for the MINMAX criterion to our toy example. We first solve the minimum rate problem stated in Eq. (4). As we did in the MINAVE case, we solve the extreme cases by inspection. First we consider the case where D_{max} converges towards infinity. Hence the smallest possible rate is obtained, which is equivalent to using the coarsest quantizers for both blocks. The resulting quantizer sequence is therefore $\{2,2\}$, which is sequence (A). The rate and distortion for this sequence are displayed in Table 2. Now we consider the case where D_{max} converges towards zero. Hence the largest possible rate is obtained, which is equivalent to using the finest quantizers for both blocks. The resulting quantizer sequence is therefore $\{1, 1\}$, which is sequence (B). The rate and distortion for this sequence are displayed in Table 2.

We now look at a more interesting case, where we need to employ dynamic programming to find the shortest path from S to T. We set D_{max} to 6, which results in the fact that all edges which are associated with a distortion greater than 6 are removed from the graph. This is true since according to Eq. (9), the rate of these edges is set to infinity, which is equivalent to removing them. The weight of the edges which are not removed are set to their rate. Hence the minimum rate problem can be solved by finding the shortest path from S to T using dynamic programming. This is shown in Fig. 4, where the boxes above the nodes contain the minimum rate up to and including that node. Again, the dashed arrows are the backpointers used to backtrack the optimal solution which is sequence (D) or $\{2, 1\}$. The rate and distortion for this sequence are displayed in Table 2.

As pointed out in the MINAVE section, there are four possible quantizer sequences and the fourth sequence is (C) or $\{1, 2\}$). Again, the rate and the distortion for this sequence are displayed in Table 2. In Fig. 6 the operational rate distortion curve is displayed. Note that we included sequence (C) for completeness, even though it does not belong to the operational rate distortion curve since sequence (A) has the same rate but a smaller distortion. So it will never be optimal to select (C) over (A). As can be seen in Fig. 6, there are no inaccessible optimal solutions.

So far we have shown that the minimum rate problem can be solved directly using a dynamic programing formulation. The minimum distortion problem formulated in Eq. (5) can be solved using the non-increasing property of the operational rate distortion curve as explained above. We go now through the exercise of finding the solution to the minimum distortion problem of Eq. (5), where the available bits R_{max} are set to 18. Since we use bisection to find the optimal D^*_{max}, we need an initial bracket for D_{max}, say $D_l = 3$ and $D_u = 10$. We then build the graph for D_l and find the shortest path which results in the sequence (B) (this can be seen from Fig. 6). Then we repeat the same for D_u. This results in sequence (A). Since the rate for (B) is higher than 18 and the rate for (A) is lower than 18, we know that the optimal D^*_{max} has to be between D_l and D_u. We then pick D_m as the average of the bracketing D's. We then build the graph for D_m and find the shortest path which then results in a certain rate and distortion. If the rate is higher than 18 then $D_l = D_m$ otherwise $D_u = D_m$. Then a new D_m is selected and the iteration is repeated until the optimal solution is found or the D bracket is small enough or a certain maximum number of iterations have been carried out. By inspecting Fig. 5 it is clear that the final solution is (D), since it uses exactly 18 bits.

7 Example

In this section we present an example to compare the MINAVE and the MINMAX approaches. The dependent image coding scheme we use for this example is the Intra frame scheme employed in TMN4 [10], which is the test model four of the H.263 standard.

Note that even though this example is quite similar to the toy example, it is important to notice that the presented theory can also be used for completely different coding schemes, as long as the assumptions stated in section 3 are satisfied. For example in [11, 4] we formulated an optimal boundary encoding scheme using the presented general theory for the MINMAX criterion.

For the example at hand, we encode the first frame of the QCIF color sequence “Mother and Daughter”. We use the TMN4 mechanism for transmitting the quantizer step sizes which is based on a modified delta modulation scheme. In TMN4, the quantizer step size of the current macro block must be within ± 2 of the quantizer step size employed for the previous macro block. Then the difference between the quantizer step sizes is entropy coded. This DPCM scheme results in a first order dependency between two consecutive blocks, since the operational rate distortion curve of the current block depends on the quantizer selected for the previous block.

First we fix the quantizer step size for all macro blocks to 10. The resulting rate ($R_{Q=10} = 18297$ bits) and distortion are listed in Table 3. Note that the mean squared error (MSE) of the luminance (Y) channel is used as the distortion measure. For both, the MINAVE and the MINMAX criterion, we solve the minimum distortion problem, where we set the maximum rate equal to the the rate TMN4 uses for a fixed quantizer of 10 ($R_{max} = R_{Q=10}$). Again, the resulting rate and distortion are listed in Table 3.

In Fig. 7 the MSE per macro block for the three implementations is shown and in Fig. 8 the corresponding quantizer selections are displayed. It is interesting to notice in Fig. 8 that there are quite a few blocks where the quantizers are the same for both optimal schemes. These blocks tend to coincide with the blocks where the MSE (see Fig. 7) is very small, i.e., blocks with no high frequency components. It is clear from Fig. 7 that the minimum maximum distortion scheme results in a more even quality for the entire frame than the minimum total distortion approach. In fact, discounting the blocks with very low MSE, the distortion profile is quite flat and very close to the minimum average distortion achieved by the MINAVE approach. In other words, the result shows that if the goal is to have almost constant distortion, which is almost as low as the smallest possible average distortion, for a given bit budget, the MINMAX criterion is an excellent choice.

8 Conclusions

We conclude this paper by comparing the two optimal bit allocation algorithms for the MINAVE and the MINMAX criterion. The MINAVE approach is based on the Lagrange multiplier method. The Lagrange multiplier method is used to transform the constrained optimization problem into a set of unconstrained optimization problems parameterized by the Lagrangian multiplier λ. These unconstrained problems are then solved optimally using dynamic programming. The optimal λ^*, which results in the solution of the original constrained problem, is then found using an iterative approach, such as bisection, where for each iteration the unconstrained problem needs to be solved. For the MINAVE approach, the minimum rate and the minimum distortion problem are both solved by the same algorithm. This is one of the main differences between the MINAVE and the MINMAX approach. For the MINMAX approach, the minimum rate problem, which is a constrained optimization problem, can be transformed into an unconstrained problem using the re-definition of the source rates. Then this unconstrained problem can be solved directly using dynamic programming. In other words, *no* iteration is necessary to solve the minimum rate problem. The minimum distortion problem is then solved using the fact that the we can find the optimal solution to the minimum rate problem, which results in a non-increasing operational rate distortion function. The solution to the minimum rate problem is also found by an iterative search for the optimal D^*_{max} using bisection. For each iteration, the minimum rate problem (i.e., the unconstrained problem) is solved using dynamic programming. While these algorithms have many similarities, they are quite different with respect to finding all optimal solutions. This cannot be guaranteed for the Lagrangian approach, since only solutions which belong to the convex hull can be found. Furthermore, while the Lagrangian multiplier method needs an iterative search, for both, the minimum rate and the minimum distortion problem, the MINMAX approach only needs an iteration for the minimum distortion problem. Hence the minimum rate problem can be solved much faster for the MINMAX approach. Ultimately, showing that the MINMAX approach has several algorithmic advantages over the MINAVE approach does not help somebody who needs to solve a MINAVE problem. We do believe however that many real world problems are better served by using a MINMAX approach than a MINAVE approach and that the MINAVE approach has only been so popular since its optimal solution has been known. Now that the optimal solution to the MINAVE criterion is known, we believe many problems should be solved using this approach. This is especially true, since both approaches are based on the same underlying assumptions. Hence every algorithm which is using the MINAVE criterion can easily be changed to use the MINMAX criterion.

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Name	λ	Sequence	Rate	Distortion
(A)	$\rightarrow \infty$	$\{2,2\}$	10	12
(B)	$\rightarrow 0$	$\{1,1\}$	19	3
(C)	1	$\{1,2\}$	13	8
(D)	N/A	$\{2,1\}$	18	7

Table 1: The rates and distortions for the toy example using the MINAVE criterion

Name	D_{max}	Sequence	Rate	Distortion
(A)	$\rightarrow \infty$	$\{2,2\}$	10	7
(B)	$\rightarrow 0$	$\{1,1\}$	19	2
(C)	N/A	$\{1,2\}$	13	7
(D)	6	$\{2,1\}$	18	5

Table 2: The rates and distortions for the toy example using the MINMAX criterion

	Rate	Distortion (MSE)			
		mean	min	max	std
Q=10	18297	27.8	0.6	66.8	17.5
min total	18431	27.1	0.6	65.0	17.3
min max	18293	29.9	0.6	46.2	15.9

Table 3: Each of the 99 macro blocks (16×16) results in a particular MSE, and the mean MSE is the mean of these 99 MSEs. The same holds for the minimum, maximum ar

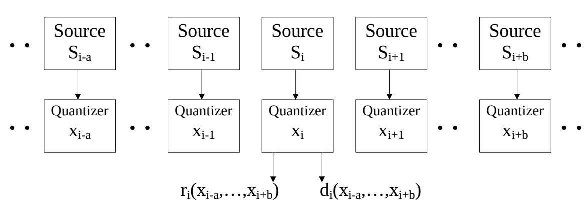

Figure 1: A dependent coding framework: the source rate and distortion depend on a neighborhood of quantizers

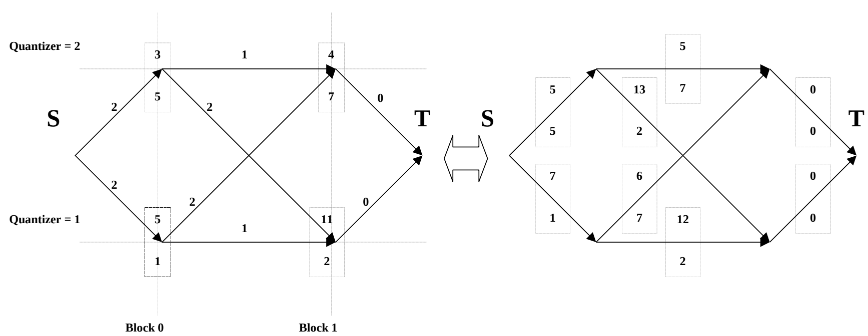

Figure 2: The graph (trellis) of the toy example. The left graph shows the rate required to switch the quantizer on top of the edges. The rate needed and distortion resulting by coding a certain block with a certain quantizer are displayed inside of the dashed boxes. The rate is above the node and the distortion below. The right graph is equivalent to the left graph, just that the all the rates and distortions are merged onto the edges.

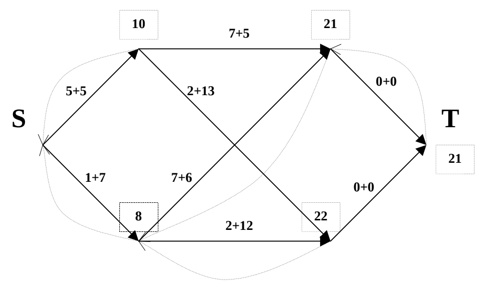

Figure 3: The graph for a λ of one. The dashed boxes contain the smallest Lagrangian cost up to and including the node. The dashed arrows are the back-pointers used to remember the optimal

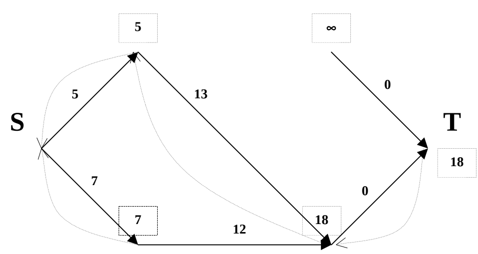

Figure 4: The graph for a D_{max} of 6. The dashed boxes contain the smallest rate up to and including the node. The dashed arrows are the backpointers used to remember the optimal quantizer sequence.

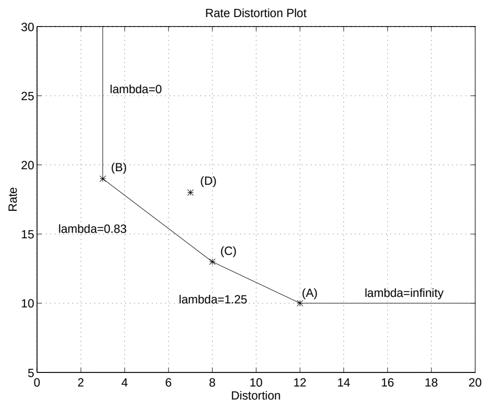

Figure 5: Operational rate distortion curve for the toy example, using the MINAVE criterion. Note the convex hull and the fact that the slope of the convex hull is related to λ as follows, slope = -1/λ.

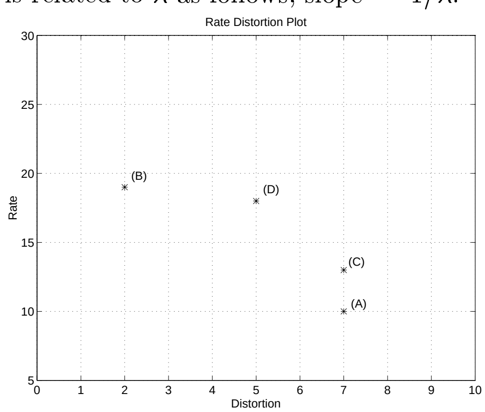

Figure 6: Operational rate distortion curve for the toy example, using the MINMAX criterion. Note that (C) is not part of the operational rate distortion curve, since (A) has the same distortion but a smaller rate. (C) is only included for completeness.

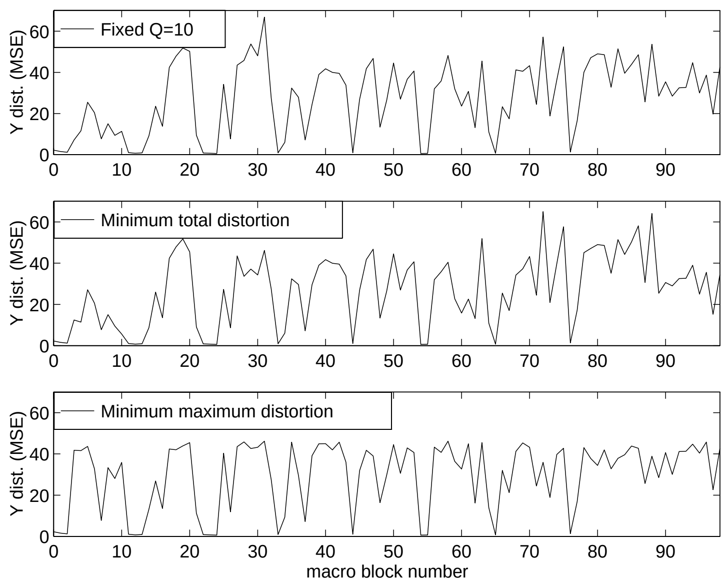

Figure 7: MSE of each macro block of the luminance channel; First row: MSE for a fixed quantizer step size of Q=10. Second row: MSE for the minimum total distortion approach. Third row: MSE for the minimum maximum distortion approach.

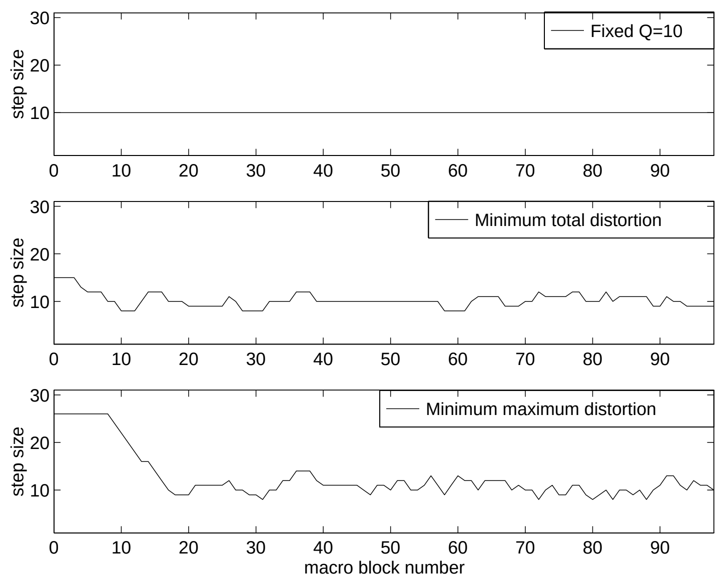

Figure 8: Macro block quantizer step sizes; First row: fixed quantizer step size of Q=10. Second row: step sizes for the minimum total distortion approach. Third row: step sizes for the minimum maximum distortion approach.